Calculus II - Day 14

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Goals for today:

- Use Pythagorean and half-angle identities to integrate functions of the form: $\int \sin^m(x)\cos^n(x)\,dx$.
- Introduce the idea of a trigonometric substitution and use it to prove the area formula for a circle.

Pythagorean Identities:

$$\sin^2(x) + \cos^2(x) = 1$$
$$1 + \cot^2(x) = \csc^2(x)$$
$$\tan^2(x) + 1 = \sec^2(x)$$

Half-Angle Formulas:

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$
$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

Easiest Case:

$$\int \sin^m(x)\cos^n(x)\,dx \quad \text{where either } m = 1 \text{ or } n = 1$$

Example: $\int \sin(2x)\cos^4(2x)\,dx$
Choose $u = \cos(2x)$ and $du = -2\sin(2x)\,dx$:

$$\int \sin(2x)\cos^4(2x)\,dx = \int u^4\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\int u^4\,du$$

$$= -\frac{1}{2}\cdot\frac{1}{5}u^5 + C = -\frac{1}{10}\cos^5(2x) + C$$

If we had $\int \sin^5(2x)\cos(2x)\,dx$, we would choose $u = \sin(2x)$ instead.

Medium Case: $m, n > 1$ but (at least) one of them is odd.
Use the Pythagorean identity:

$$\sin^{2k+1}(x) = \sin^{2k}(x)\cdot\sin(x) = (\sin^2(x))^k\sin(x) = (1-\cos^2(x))^k\sin(x)$$

$$\cos^{2k+1}(x) = \cos(x)\cdot(1-\sin^2(x))^k$$

Example: $\int \sin^3(x)\cos^4(x)\,dx$
Pick the function with the even power to be u:

$$\sin^3(x) = (1-\cos^2(x))\sin(x)$$

$$= \int (1-\cos^2(x))\sin(x)\cos^4(x)\,dx$$

Choose $u = \cos(x)$, so $du = -\sin(x)\,dx$:

$$= \int (1-u^2)u^4(-du) = -\int (u^4 - u^6)\,du$$

$$= \int (u^6 - u^4)\,du = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C$$

$$= \frac{1}{7}\cos^7(x) - \frac{1}{5}\cos^5(x) + C$$

Check:

$$\frac{d}{dx}\left(\frac{1}{7}\cos^7(x) - \frac{1}{5}\cos^5(x) + C\right)$$

$$= \frac{1}{7}\cdot 7\cos^6(x)\cdot(-\sin(x)) - \frac{1}{5}\cdot 5\cos^4(x)\cdot(-\sin(x))$$

$$= -\cos^6(x)\sin(x) + \cos^4(x)\sin(x)$$

$$= \sin(x)\cos^4(x)(-\cos^2(x)+1)$$

$$= \sin^3(x)\cos^4(x)$$

Example to Try:

$$\int \cos^5(x)\,dx$$

Rewrite using the identity for odd powers:

$$\cos^5(x) = \cos^4(x)\cos(x) = (1-\sin^2(x))^2\cos(x)$$

Now, let $u = \sin(x)$, so $du = \cos(x)\,dx$:

$$= \int (1-u^2)^2\,du$$

$$= \int (1-2u^2+u^4)\,du$$

$$= \int 1\,du - 2\int u^2\,du + \int u^4\,du$$

Integrate each term:

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

Substitute $u = \sin(x)$ back:

$$= \sin(x) - \frac{2}{3}\sin^3(x) + \frac{1}{5}\sin^5(x) + C$$

$$\boxed{\int \cos^5(x)\,dx = \sin(x) - \frac{2}{3}\sin^3(x) + \frac{1}{5}\sin^5(x) + C}$$

Hard case: even powers of both sine and cosine

Recall:

$$\int \tan(x)\,dx = \ln|\sec(x)| + C, \quad \int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$

Example:

$$\int \tan^3(x)\sec^4(x)\,dx$$

Using the identity

$$\tan^2(x) = \sec^2(x) - 1:$$

$$= \int (\sec^2(x)-1)\tan(x)\sec^4(x)\,dx$$

$$= \int (\sec^2(x)-1)\sec^3(x)\sec(x)\tan(x)\,dx$$

$$\text{Let } u = \sec(x), \text{ then } du = \sec(x)\tan(x)\,dx:$$

$$= \int (u^2 - 1)u^3 \, du$$

$$= \int (u^5 - u^3) \, du$$

$$= \frac{1}{6}u^6 - \frac{1}{4}u^4 + C$$

Substitute back

$$u = \sec(x) :$$

$$= \frac{1}{6}\sec^6(x) - \frac{1}{4}\sec^4(x) + C$$

Hard case:

$$\int_0^{\pi/2} \sin^2(x) \, dx$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} = \frac{1}{2} - \frac{\cos(2x)}{2}$$

$$= \int_0^{\pi/2} \left(\frac{1}{2} - \frac{\cos(2x)}{2} \right) dx = \int_0^{\pi/2} \frac{1}{2} \, dx - \int_0^{\pi/2} \frac{\cos(2x)}{2} \, dx$$

$$= \frac{\pi}{4} - \int_0^{\pi/2} \frac{\cos(2x)}{2} \, dx$$

$$\text{Let } u = 2x, \text{ then } du = 2 \, dx \Rightarrow dx = \frac{du}{2}$$

$$u(0) = 0, \quad u\left(\frac{\pi}{2}\right) = \pi$$

$$= \frac{\pi}{4} - \int_0^{\pi} \frac{\cos(u)}{4} \, du$$

$$= \frac{\pi}{4} - \frac{1}{4}\sin(u)\Big|_0^{\pi}$$

$$= \frac{\pi}{4} - \frac{1}{4}(\sin(\pi) - \sin(0)) = \frac{\pi}{4} - \frac{1}{4}(0 - 0) = \frac{\pi}{4}$$

Harder case:

$$\int \sin^2(x)\cos^4(x) \, dx$$

$$= \int \sin^2(x) \left(\cos^2(x)\right)^2 \, dx$$

$$= \int \left(\frac{1 - \cos(2x)}{2}\right) \left(\frac{1 + \cos(2x)}{2}\right)^2 \, dx$$

$$= \frac{1}{8} \int (1 - \cos(2x))(1 + 2\cos(2x) + \cos^2(2x)) \, dx$$

$$= \frac{1}{8} \int \left(1 + 2\cos(2x) + \cos^2(2x) - \cos(2x) - 2\cos^2(2x) - \cos^3(2x)\right) \, dx$$

$$= \frac{1}{8}\int \left(1 + \cos(2x) - \cos^2(2x) - \cos^2(2x)\right)\, dx$$

$$= \frac{1}{8}\int dx + \frac{1}{8}\int \cos(2x)\, dx - \frac{1}{8}\int \cos^2(2x)\, dx - \frac{1}{8}\int \cos^3(2x)\, dx$$

$$= \frac{1}{8}x + \frac{1}{16}\sin(2x) - \frac{1}{8}\int \frac{1+\cos(4x)}{2}\, dx - \frac{1}{8}\int (1 - \sin^2(2x))\cos(2x)\, dx$$

$$\frac{1}{8}x + \frac{1}{16}\sin(2x) - \frac{1}{8}\int \frac{1+\cos(4x)}{2}\, dx - \frac{1}{8}\int (1 - \sin^2(2x))\cos(2x)\, dx$$

$$-\frac{1}{8}\int (1 - \sin^2(2x))\cos(2x)\, dx$$

$$\text{Let } u = \sin(2x), \text{ so } du = 2\cos(2x)\, dx \Rightarrow dx = \frac{du}{2\cos(2x)}$$

$$= -\frac{1}{8}\int \left(1 - u^2\right)\frac{1}{2}\, du$$

$$= \frac{1}{8}x + \frac{1}{16}\sin(2x) - \frac{1}{16}x - \frac{1}{64}\sin(4x) - \frac{1}{16}u + \frac{1}{48}u^3 + C$$

$$\text{Substitute back } u = \sin(2x):$$

$$= \frac{1}{16}x + \frac{1}{16}\sin(2x) - \frac{1}{64}\sin(4x) - \frac{1}{16}\sin(2x) + \frac{1}{48}\sin^3(2x) + C$$

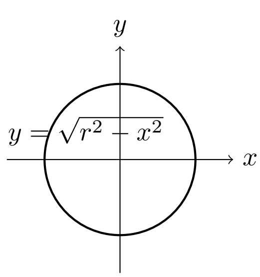
$$= \frac{1}{16}x - \frac{1}{64}\sin(4x) + \frac{1}{48}\sin^3(2x) + C$$

Integrals involving $a^2 - x^2$, $a^2 + x^2$, or $x^2 - a^2$

Example: A circle with radius r has the formula:

$$x^2 + y^2 = r^2$$

$$y = \pm\sqrt{r^2 - x^2}$$



$$\text{Area: } 2\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$$

How do we integrate this? We make an "inverse substitution":

$$x = r\sin(\theta)$$

$$\frac{dx}{d\theta} = r\cos(\theta) \Rightarrow dx = r\cos(\theta)\,d\theta$$

When x = r, what is θ?

$$r = r\sin(\theta) \Rightarrow \sin(\theta) = 1 \Rightarrow \theta = \arcsin(1) = \frac{\pi}{2}$$

When x = -r, what is θ?

$$-r = r\sin(\theta) \Rightarrow \sin(\theta) = -1 \Rightarrow \theta = \arcsin(-1) = -\frac{\pi}{2}$$

$$2\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = 2\int_{-\pi/2}^{\pi/2} \sqrt{r^2 - r^2\sin^2(\theta)} \cdot r\cos(\theta)\,d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} r\cos(\theta) \cdot r\cos(\theta)\,d\theta = 2r^2 \int_{-\pi/2}^{\pi/2} \cos^2(\theta)\,d\theta$$

$$= 2r^2 \int_{-\pi/2}^{\pi/2} \frac{1 + \cos(2\theta)}{2}\,d\theta$$

$$= 2r^2 \left(\frac{1}{2}\theta + \frac{\sin(2\theta)}{4}\right)\Big|_{-\pi/2}^{\pi/2}$$

$$= 2r^2 \cdot \frac{1}{2} \cdot \pi = \pi r^2$$